

§ 15.4/15.5 Triple Integrals

①

• The triple integral:

$$\iiint_D f(x, y, z) dV$$

We'd like to call this "the 4-D volume under the graph of f 'above' the 3-D volume D ".

But we can't visualize 4-D volumes —

So we view it as

"A sum of weighted 3-volumes"

• For this - define the integral ^②
as the limit of a 3-D

Riemann Sum: $\iiint_D f(x, y, z) dV$

weighted 3-volume

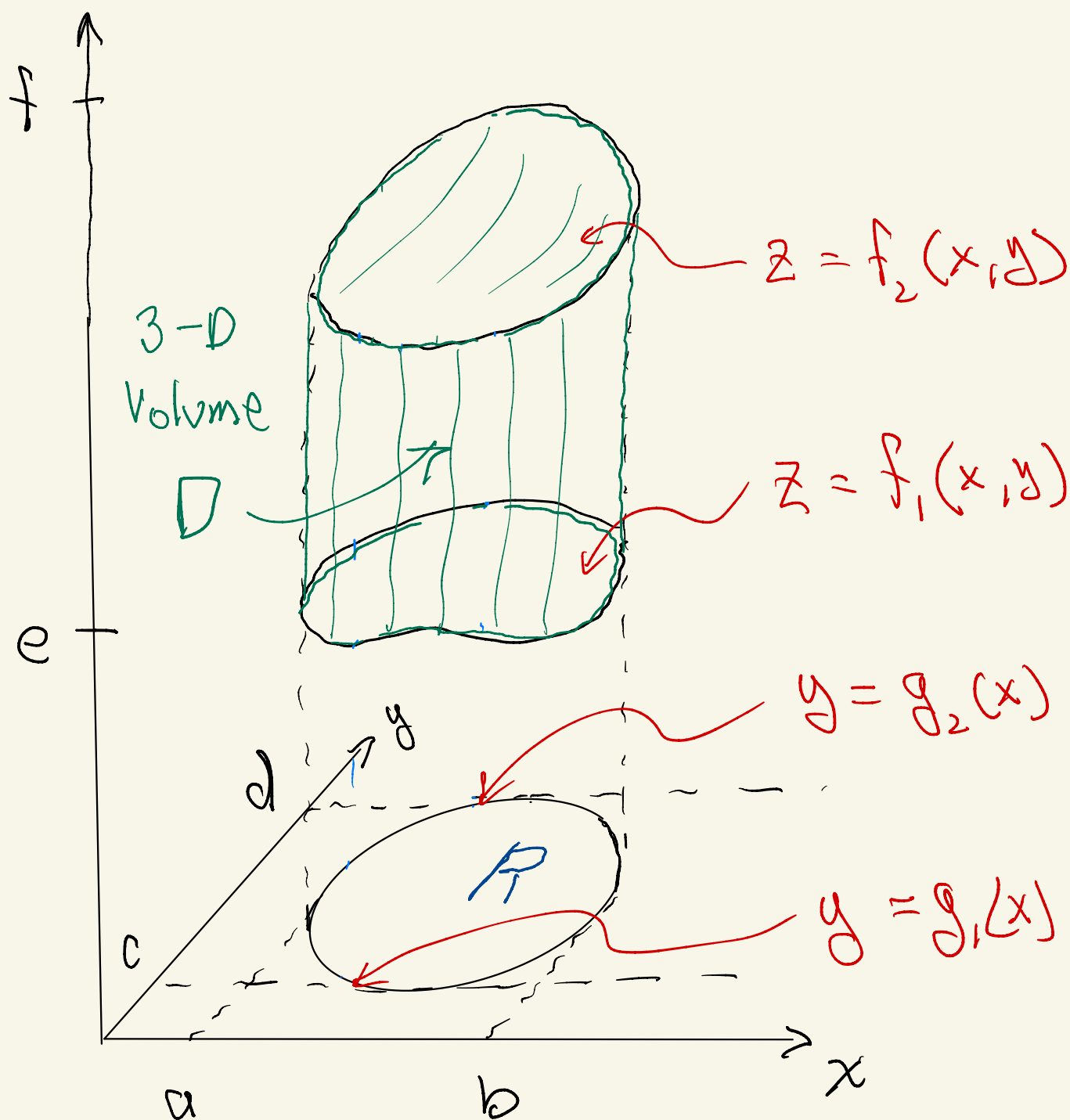
$$= \lim_{N \rightarrow \infty} \sum_{(x_i, y_j, z_k) \in D} f(x_i, y_j, z_k) \underbrace{\Delta x \Delta y \Delta z}_{\Delta V_{ijk}}$$

3-D Riemann Sum

- Approx by 4-D rectangles
and view the integral as
"a sum of weighted 3-volumes"

• The basic Picture :

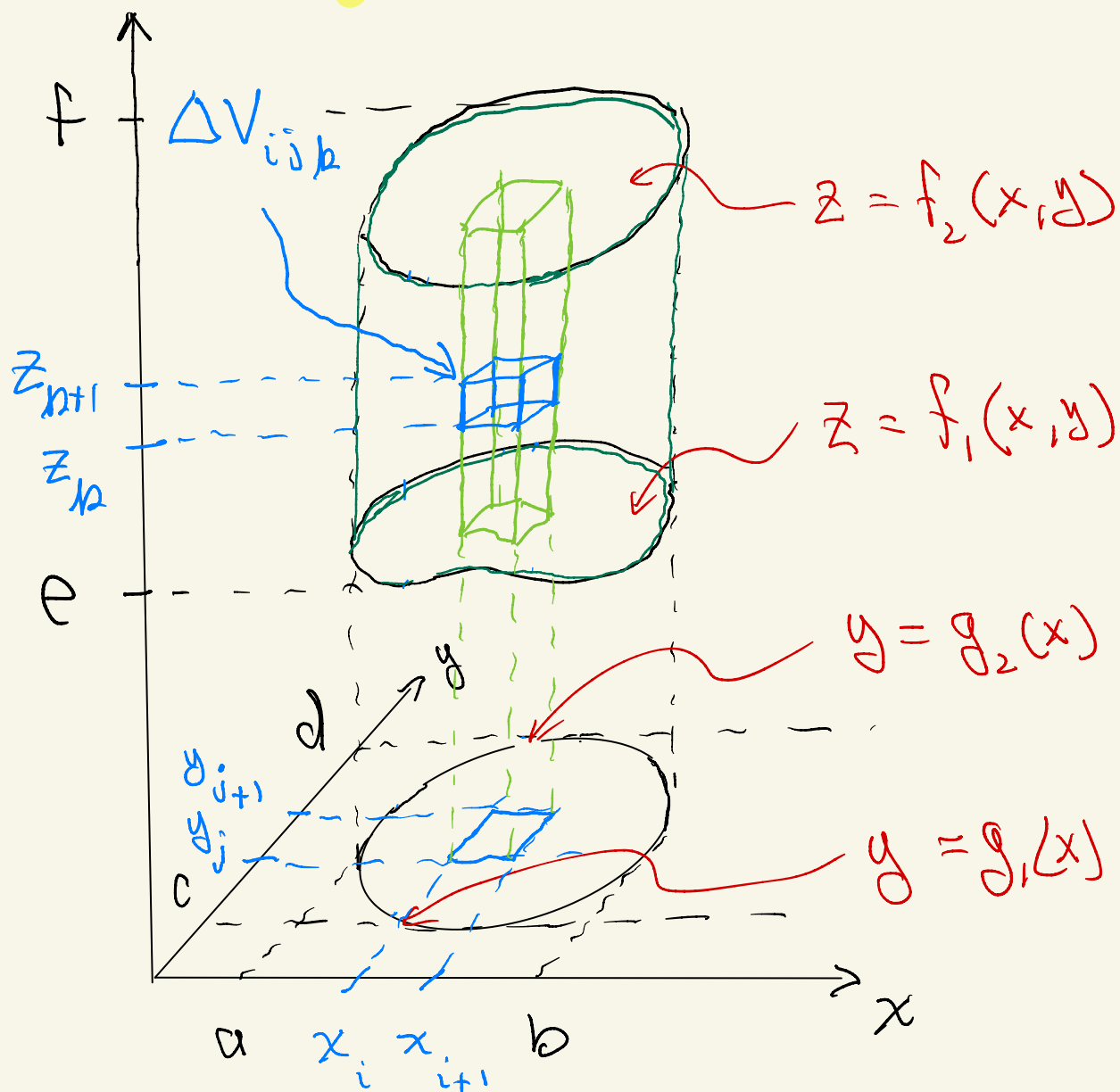
③



$$D \subseteq [a, b] \times [c, d] \times [e, f]$$

3-D Riemann Sum

(4)



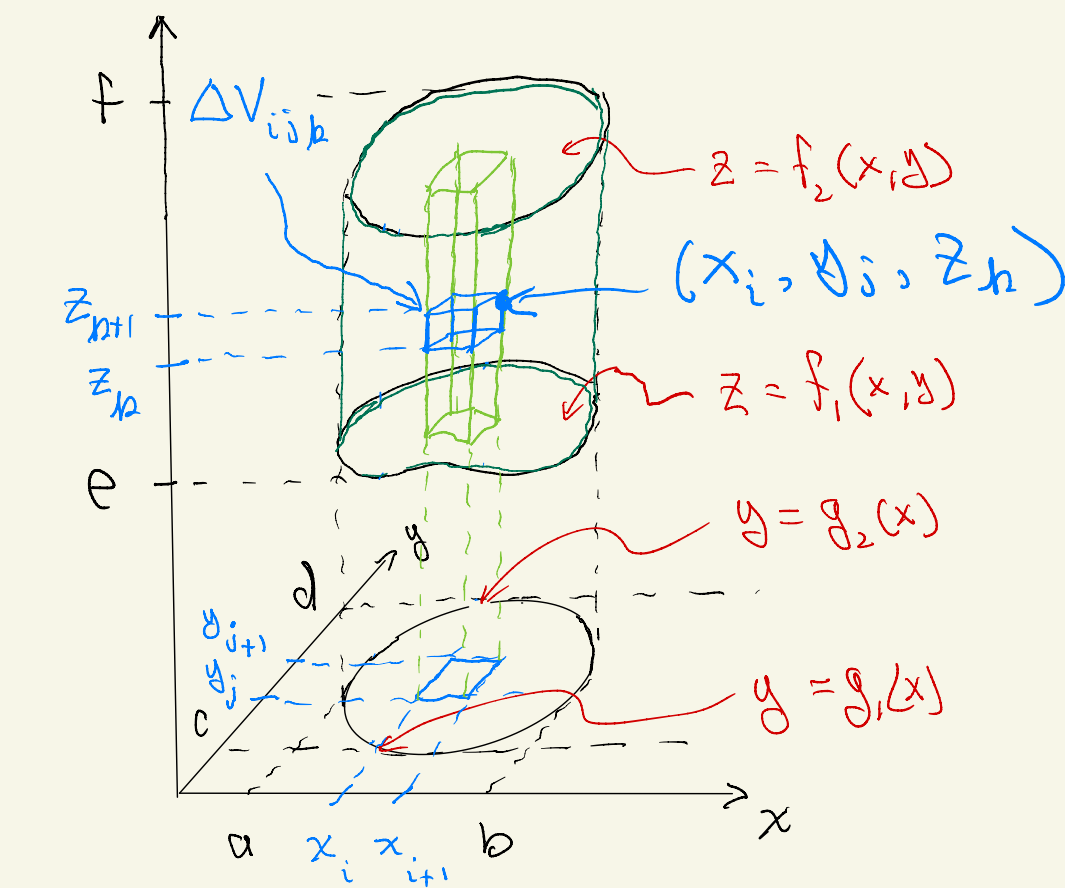
$$a = x_0 < x_1 < x_2 < \dots < x_i < \dots < x = b$$

$$c = y_0 < y_1 < y_2 < \dots < y_j < \dots < y_N = d$$

$$e = z_0 < z_1 < \dots < z_N = f$$

$$\Delta x = \frac{b-a}{N}, \quad \Delta y = \frac{d-c}{N}, \quad \Delta z = \frac{f-e}{N}$$

(5)



$$\Delta x = \frac{b-a}{N}, \Delta y = \frac{d-c}{N}, \Delta z = \frac{f-e}{N}$$

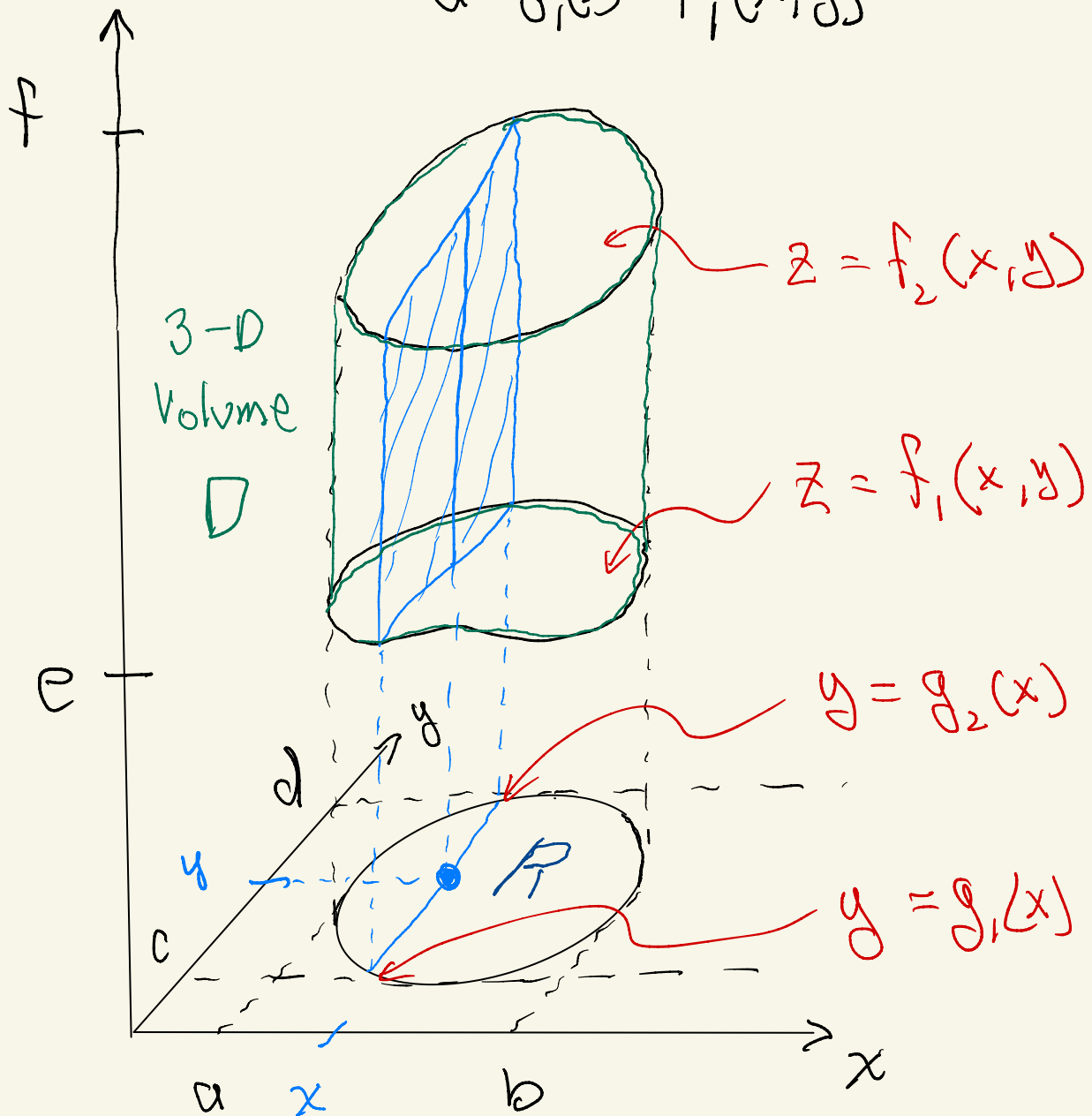
• Definition of 3-D integral as a Riemann Sum:

$$\iiint_D f(x, y, z) dv = \lim_{N \rightarrow \infty} \sum_{(x_i, y_j, z_n) \in D} f(x_i, y_j, z_n) \Delta x \Delta y \Delta z$$

Approximation by Riemann Sum

- Iterate the Integral to get exact value (6)

$$\iiint_D f(x, y, z) dv = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x, y)}^{f_2(x, y)} f(x, y, z) dz dy dx$$



■ § 15.5 Same idea to get 7
Area / Mass / Center of Mass / Moments of Inertia
formulas from Riemann Sum?

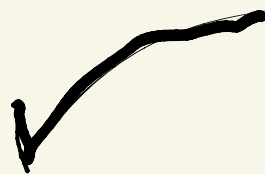
Assume D is a 3-D object
with density $\delta(x, y, z) = \frac{\text{mass}}{\text{volume}}$

• Volume: Take $\delta = 1$

$$\text{Vol}(D) = \lim_{N \rightarrow \infty} \sum_{(x_i, y_i, z_i) \in D} 1 \cdot \Delta x \Delta y \Delta z$$

$$= \lim_{N \rightarrow \infty} \sum_D \Delta V_{i,n}$$

$$V = \iiint_D dv$$



- Mass:

⑧

$$\text{Mass}(D) = \lim_{N \rightarrow \infty} \sum_{(x_i, y_i, z_i) \in D} \delta(x_i, y_i, z_i) \Delta x \Delta y \Delta z$$

$$\frac{\Delta \text{Mass}}{\Delta \text{Vol}} \cdot \text{Vol}$$

$$= \lim_{N \rightarrow \infty} \sum_D \Delta M_{i,j,k}$$

$$M = \iiint_D \delta(x, y, z) dV$$

• Center of Mass (Similarly)

⑨

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

$$M_{yz} = \iiint_D x \delta(x, y, z) dV$$

Distance to
yz-plane = x

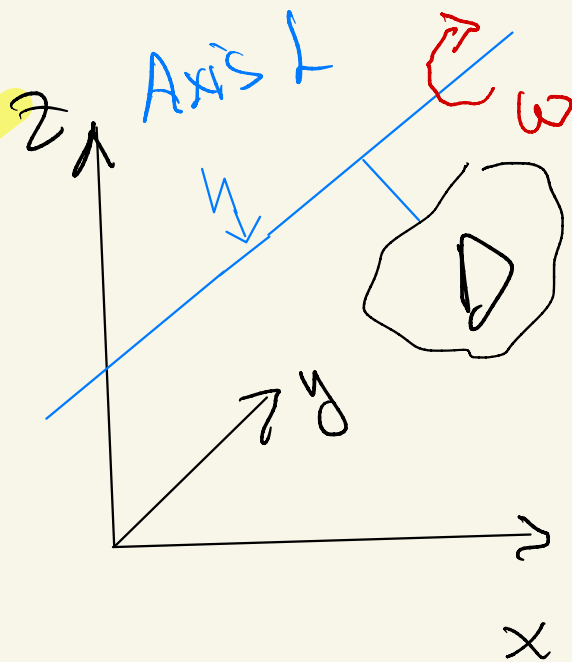
$$M_{xz} = \iiint_D y \delta(x, y, z) dV$$

$$M_{xy} = \iiint_D z \delta(x, y, z) dV$$

• Kinetic Energy of Rotation

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Q: What is the KE of rotation about axis L at angular velocity



$$\omega = \frac{d\theta}{dt} \quad ?$$

$$KE = \lim_{N \rightarrow \infty} \sum_D \frac{1}{2} \Delta M_{i\Delta n} (\omega r_{i\Delta n})^2$$

$$= \frac{1}{2} \omega^2 \lim_{N \rightarrow \infty} \sum_D r_{i\Delta n}^2 \delta(x, y, z) \Delta x \Delta y \Delta z$$

$$= \frac{1}{2} \omega^2 \iiint_D r(x, y, z)^2 \delta(x, y, z) dV$$

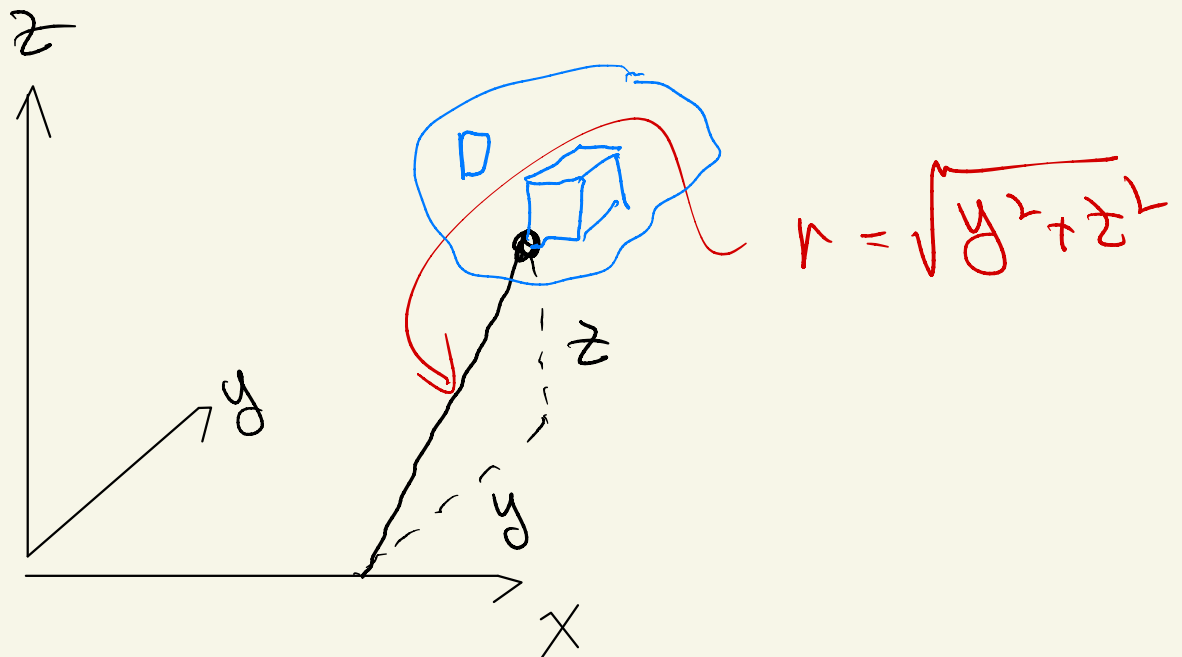
$$I_L = \frac{1}{2} I_L \omega^2$$

Conclude:

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$$KE = \frac{1}{2} I_L \omega^2$$

Fig $I_x = \iiint_D \underbrace{(y^2 + z^2)}_{\text{distance to } x\text{-axis}} dV$



Radius of Gyration R
is defined by the equation

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$$\frac{1}{2} M \omega^2 R^2 = \frac{1}{2} I_L \omega^2$$

Solve for R :

$$\cancel{\frac{1}{2}} M \cancel{\omega^2} R^2 = \cancel{\frac{1}{2}} I_L \cancel{\omega^2}$$

$$R^2 = \frac{I_L}{M}$$

$$R = \sqrt{\frac{I_L}{M}}$$

Examples

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① Find the volume of the tetrahedron D bounded from above by the plane thru $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, and on the sides by the xy -plane, yz -plane, xz -plane

Assume: $a, b, c > 0$ positive
n.v.

- Draw the region

Eq for plane: $z = Ax + By + C$

$$(0, 0, c) \Rightarrow c = 0 + 0 + C$$

$$C = c$$

$$(0, b, 0) \Rightarrow 0 = 0 + Bb + c$$

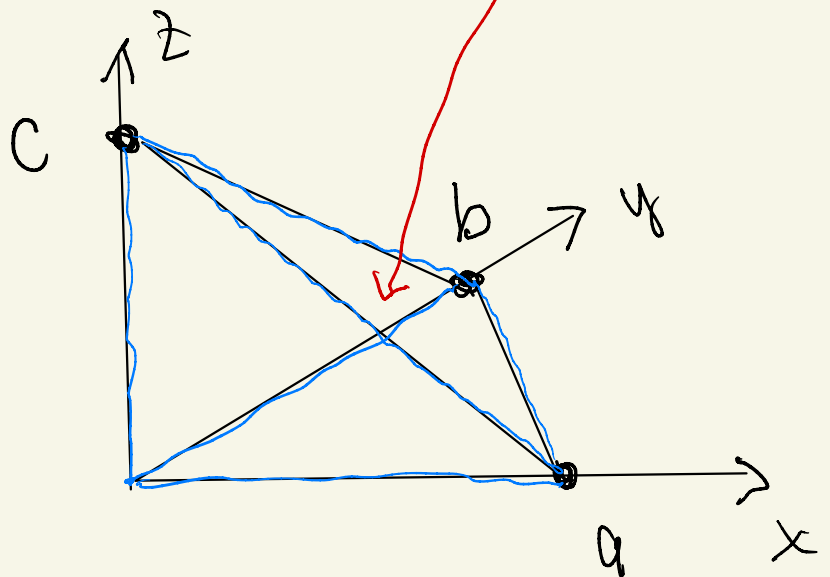
$$B = -\frac{c}{b}$$

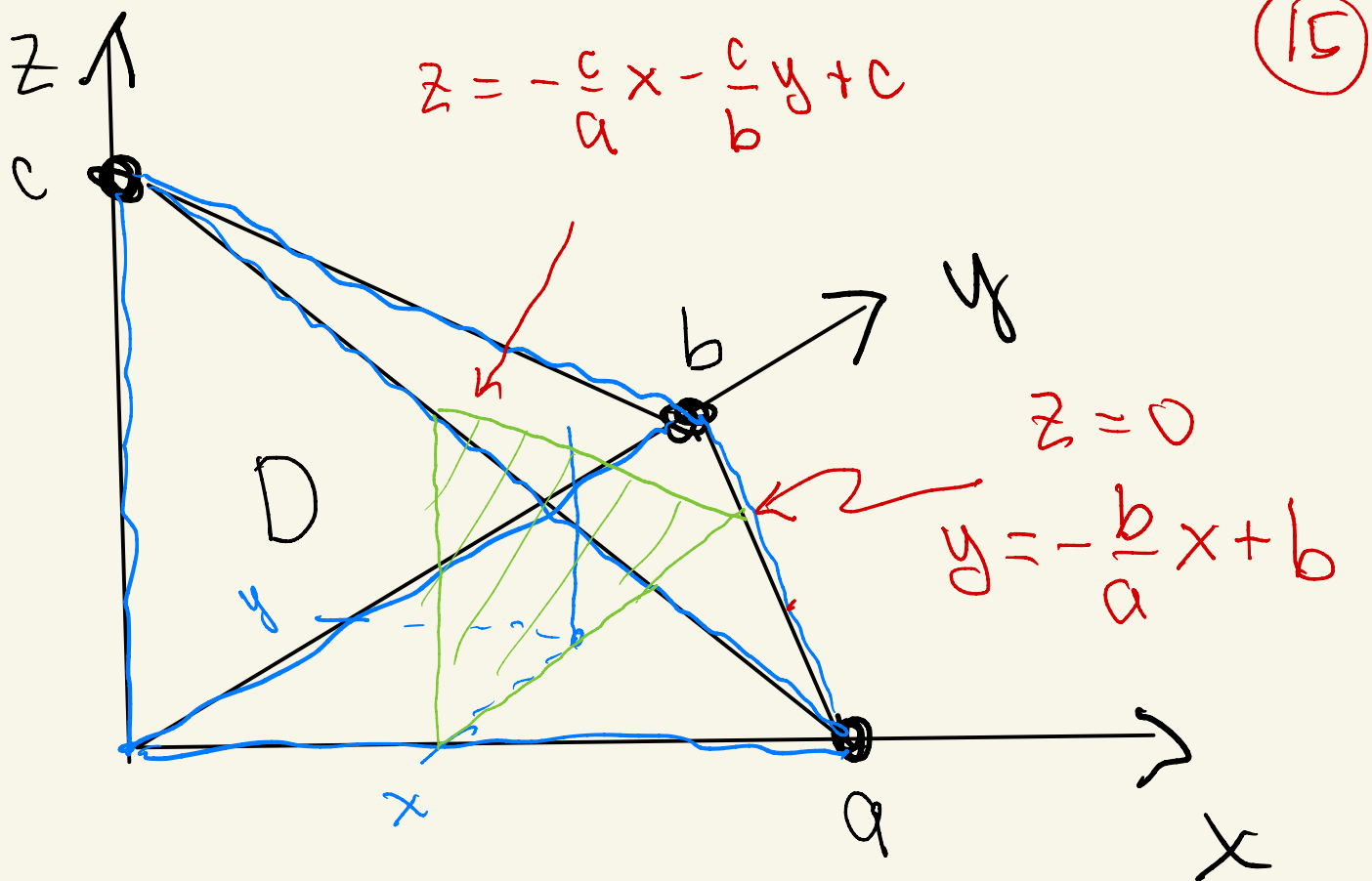
$$(a, 0, 0) \Rightarrow 0 = Aa + c$$

$$A = -\frac{c}{a}$$

$$z = -\frac{c}{a}x - \frac{c}{b}y + c$$

- From the picture we get limits of integration





$Vol = \iiint_D 1 \cdot dV$

$= \int_0^a \int_0^{-\frac{b}{a}x+b} \int_0^{-\frac{c}{a}x - \frac{c}{b}y + c} dz \, dy \, dx$

fix (x,y)

fix x

Evaluate by 3 Math 21B integrals

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$$\text{Vol}(D) = \int_0^a \int_0^{-\frac{b}{a}x+b} \int_0^{-\frac{c}{a}x - \frac{c}{b}y + c} dz \, dy \, dx$$

$$\left[z \right]_0^* = -\frac{c}{a}x - \frac{c}{b}y + c$$

$$= \int_0^a \int_0^{-\frac{b}{a}x+b} \left(-\frac{c}{a}x - \frac{c}{b}y + c \right) dy \, dx$$

$$\left[-\frac{c}{a}xy - \frac{c}{b} \frac{y^2}{2} + cy \right]_0^{-\frac{b}{a}x+b}$$

$$= -\frac{c}{a}x \left(-\frac{b}{a}x + b \right) - \frac{c}{b} \frac{1}{2} \left(-\frac{b}{a}x + b \right)^2$$

$$+ c \left(-\frac{b}{a}x + b \right)$$

Have:

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$$= \int_0^a \int_0^{-\frac{b}{a}x+b} -\frac{c}{a}x - \frac{c}{b}y + c \, dy \, dx$$

$$-\frac{c}{a}x\left(-\frac{b}{a}x+b\right) - \frac{c}{b} \frac{1}{2} \left(-\frac{b}{a}x+b\right)^2 + c\left(-\frac{b}{a}x+b\right)$$

$$= \int_0^a \frac{cb}{a^2} x^2 - \frac{cb}{a} x - \frac{cb}{a} x + bc$$

$$- \frac{1}{2} \frac{c}{b} \left(\frac{b^2}{a^2} x^2 - 2 \frac{b^2}{a} x + b^2 \right) dx$$

$$= \int_0^a \frac{cb}{a^2} x^2 - 2 \frac{cb}{a} x + bc$$

$$- \frac{1}{2} \frac{cb}{a^2} x^2 + \frac{cb}{a} x - \frac{1}{2} cb$$

$$= \int_0^a \frac{1}{2} \frac{bc}{a^2} x^2 - \frac{bc}{a} x + \frac{1}{2} bc \, dx$$

$$= \frac{bc}{a} \int_0^a \frac{1}{2} \frac{x^2}{a} - x + \frac{1}{2} a \, dx$$

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$$= \int_0^a \left[\frac{1}{2} \frac{bc}{a^2} x^2 - \frac{bc}{a} x + \frac{1}{2} bc \right] dx$$

$$= \frac{bc}{a} \int_0^a \left[\frac{1}{2} \frac{x^2}{a} - x + \frac{1}{2} a \right] dx$$

$$= \frac{bc}{a} \left[\frac{1}{6} \frac{x^3}{a} - \frac{x^2}{2} + \frac{1}{2} ax \right]_0^a$$

$$= \frac{bc}{\cancel{a}} \left[\frac{1}{6} \frac{\overset{1}{\cancel{a}}^2 \overset{3}{a}}{\cancel{a}} - \frac{\overset{1}{\cancel{a}}^2}{2} + \frac{1}{2} \overset{1}{\cancel{a}}^2 \right]$$

$$= bc \left[\frac{1}{6} a - \frac{1}{2} \cancel{a} + \frac{1}{2} \cancel{a} \right]$$

$$= \frac{1}{6} abc$$

Volume of Tetrahedron $V = \frac{abc}{6}$

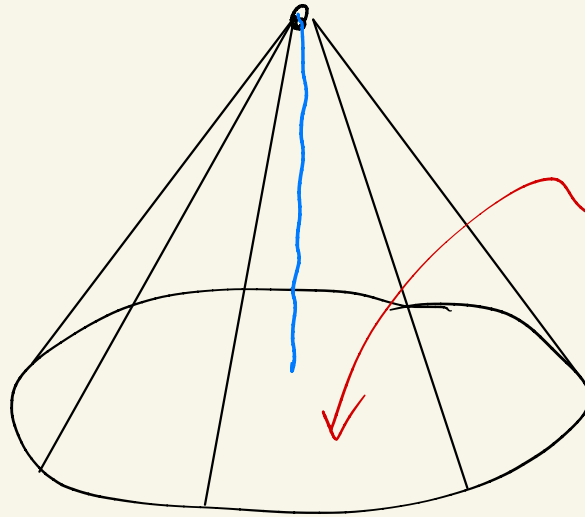




Check:

The volume of a cone:

height = h



Any area A

$$\text{Volume} = \frac{1}{3} Ah$$

For Tetrahedron -

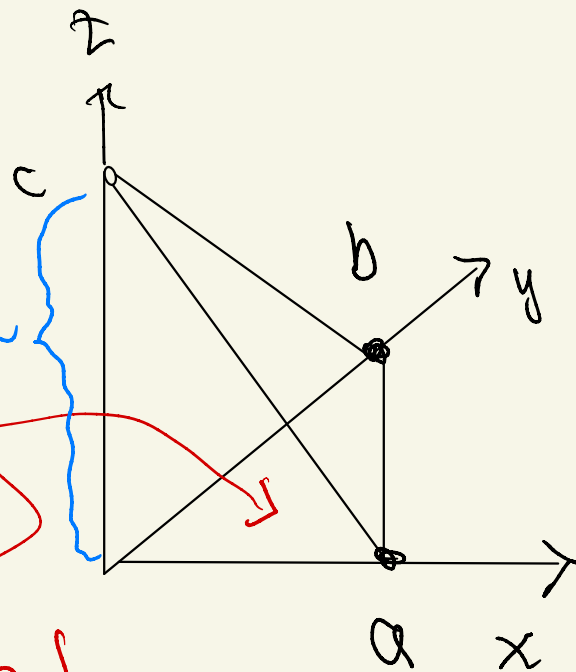
$$\text{Vol} = \frac{1}{3} Ah$$

$$= \frac{1}{3} \left(\frac{1}{2} ab \right) c = \frac{abc}{3}$$

Area

$$\text{base} = \frac{1}{2} ab$$

height c



Main Take Away: The essence (20) of integration in 3-dimensions is captured in the following picture

